Invariant Subspaces for Integration and Differentiation on Convex Domains

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Abstract. For a conformal map g of the unit disc U onto any bounded convex domain Ω we determine the invariant subspaces for the integration operator $T_g f = \int_0^z f g'$ on the Hardy space H^p, by means of a Fock space model. Under some additional regularity assumptions on $\partial\Omega$ we also treat integration from a boundary point of U, and we determine the residual subspaces (those invariant subspaces on which the operator has empty spectrum) for the left inverse $D_g f = f'/g'$ of T_g on the smooth disc algebra A^∞. The latter result is new even for $\Omega = U$.

1. Introduction

The aim of this paper is to provide partial answers to the following two questions.

Problem A. Let U be the unit disc, fix $a \in \bar{U}$, let g be holomorphic in U, and suppose the Pommerenke operator

$$(T_g f)(z) = \int_a^z f(\zeta) g'(\zeta)\, d\zeta \tag{1}$$

is well defined and bounded on the Hardy space H^p. What are its invariant subspaces?

Problem B. Let $A^\infty = \{f : D^n f \in A \text{ for } n = 0, 1, 2, \dots\}$, where $(Df)(z) = f'(z)$ and A is the disc algebra. The topology of A^∞ is that of uniform convergence in all derivatives. What are the invariant subspaces for D on A^∞?

By an *invariant subspace* for a continuous linear operator S on a topological vector space X, we mean a closed subspace $Y \subset X$ such that $SY \subset Y$ and $Y \neq 0$, $Y \neq X$.

When $a \in T$ (the unit circle), the integral (1) is to be interpreted as the value at z of the unique antiderivative of fg' which has limit 0 at a. This makes sense for example if $fg' \in H^1$, for then $\int_0^z fg' \in A$ by Hardy's inequality.

The operators T_g of Problem A have attracted much interest in the recent decades, motivated originally by Pommerenke's observation [19] that they lead to a very simple proof of the analytic John-Nirenberg inequality. Necessary and sufficient conditions for their boundedness and compactness have been determined on a wide range of spaces, including those of Hardy [4, 9, 19, 21], Bergman [10], and Dirichlet [14]. The spectrum of T_g is also quite well understood, at least on Hardy and Bergman spaces [5, 8, 15]. In contrast, almost nothing is known about invariant subspaces beyond the simplest case of $g(z) = z$, which was solved by Aleman and Korenblum [7] in 2008.

Problem B was posed to me by Alex Bergman. Its real-variable analogue, concerning ordinary differentiation on $C^\infty(0, 1)$, has been studied by several authors. Aleman and Korenblum [6] have determined the *residual subspaces*, that is, those invariant subspaces on which the operator has empty spectrum; the corresponding spectral synthesis problem was subsequently studied by Aleman, Baranov, and Belov [1] and later solved very completely by Baranov and Belov [11].

Recently, Aleman and Bergman [3] have for a large class of pairs of operators (D, V) where V is compact with $\sigma(V) = \{0\}$ and $DV = I$ obtained a description of the

invariant subspaces for V and the residual subspaces for D, generalizing both the result of [6] and the classical characterization of the Volterra-invariant subspaces of L^2. The operators of Problems A and B are interesting examples of ones to which the techniques of [3] are difficult to apply, as they rely heavily on the existence of an orthonormal basis among the eigenvectors e_λ of D.

The general trick which is key to the arguments of both [3], [6], and [7], and which will also be the foundation of our approaches to Problems A and B, is the conversion of questions of V-invariance and D-residuality into questions of near invariance (the permission to divide out zeros) in a space X of entire functions; X is the range of the generalized Fourier transform $\hat{x}(\lambda) = \langle e_\lambda, x\rangle$ acting on the dual of the original space. In the case of Problem A this transform is injective only if $\{e^{\lambda g} : \lambda \in \mathbb{C}\}$ spans a dense subspace of H^p, and we must therefore restrict our attention to univalent g.

The hypotheses of [3] are chosen in such a way that X becomes a de Branges space. Near invariance is well understood in these spaces, thanks to de Branges' celebrated ordering theorem. Our Problem B instead gives rise to a Fock-type space, and the same goes for Problem A if $g(U)$ is assumed to be bounded and convex. While near invariance is in general quite poorly understood in Fock-type spaces, a recent result on backward-shift invariance due to Aleman, Baranov, Belov, and Hedenmalm [2] has allowed us to prove the following.

Theorem A.1. *Suppose $0 < p < \infty$ and g is a conformal map of U onto a bounded convex domain. Fix $a \in U$ and put*

$$B_a(z) = \frac{z-a}{1-\bar{a}z}.$$

Then

(*a*) *T_g is a compact operator $H^p \to H^p$ with $\sigma(T_g) = \{0\}$.*

(*b*) *$B_a^n H^p$ is T_g-invariant for $n = 1, 2, 3, \ldots$.*

(*c*) *No other subspace of H^p is T_g-invariant.*

It is obvious that (*c*) is false for any g which is not conformal; consider the closure of the polynomials in g.

For Problem B and for the case $a \in T$ of Problem A we will instead resort to a weak but very general result about near invariance: Hadamard's factorization theorem. The technique, which was also used in [7], is the following.

Suppose X is any topological vector space of entire functions of exponential type. A closed subspace $N \subset X$ ($N \neq 0$, $N \neq X$) is said to be *nearly invariant* if $F(z)/(z-\lambda) \in N$ whenever $F \in N$ and $F(\lambda) = 0$. (Thus we do not permit N to have common zeros, unlike in some literature.) For any $F \in X$, logarithmic differentiation of the Hadamard expansion

$$F(z) = e^{b+cz} z^m \prod_{n=1}^{\infty} (1 - z/\lambda_n) e^{z/\lambda_n} \tag{2}$$

leads to a locally uniformly convergent series

$$F'(z) = cF(z) + m\frac{F(z)}{z} + \sum_{n=1}^{\infty} \frac{zF(z)}{\lambda_n(z-\lambda_n)}. \tag{3}$$

If $F \in N$ and N is nearly invariant, all terms of (3) belong to N. Thus, we can draw the useful conclusion that $F' \in N$, provided that the series converges in the topology

of X. In the case of Problem A this convergence turns out to be related to the behavior of the quantity

$$d_g(\lambda) = \inf\{\|R_\lambda - S\| : \mathscr{R}(S) \subset \mathscr{C}e^{\lambda g}\} \tag{4}$$

for large $|\lambda|$. The norm in (4) is the H^p operator norm, $\mathscr{R}(S)$ denotes the range of the operator S, and

$$R_\lambda = T_g(I - \lambda T_g)^{-1}. \tag{5}$$

Since this technique does not rely on X being a Fock-type space it could potentially be used to extend Theorem A.1 to more general conformal maps g, but the author has been unable to do this — the method of estimating $d_g(\lambda)$ presented in Section 5 breaks down completely if $g(U)$ is not convex. We do however obtain the following two results.

Theorem A.2. *Suppose $1 \le p < \infty$ and g is a conformal map of U onto a Jordan domain whose boundary is C^2 with positive curvature. Fix $a \in T$ and put*

$$S_a^t(z) = \exp\left\{-t\frac{a+z}{a-z}\right\}.$$

Then

(*a*) *T_g is a compact operator $H^p \to H^p$ with $\sigma(T_g) = \{0\}$.*

(*b*) *$S_a^t H^p$ is T_g-invariant for any $t > 0$.*

(*c*) *No other subspace of H^p is T_g-invariant.*

Explicitly, the boundary curve is to have a parametrization $\Gamma \in C^2$ with $|\Gamma'| = 1$ and $|\Gamma''| > 0$. Note that this implies convexity.

By the spectrum $\sigma(\Theta)$ of an inner function $\Theta = cBS$, $|c| = 1$, we will mean the closure of $Z(B) \cup \operatorname{supp}\mu$, where $Z(B)$ is the zero set of the Blaschke product B and

$$S(z) = \exp\left\{-\int_T \frac{w+z}{w-z}\,d\mu(w)\right\}.$$

A compact set $K \subset T$ of Lebesgue measure 0 is called a *Carleson set* if the lengths δ_i of the arcs forming the complement of K satisfy $\sum \delta_i \log \delta_i > -\infty$. It is well known (see e.g. [22]) that K is a Carleson set if and only if the space

$$A_K^\infty = \{f \in A^\infty : D^n f = 0 \text{ on } K \text{ for } n = 0, 1, 2, \dots\}$$

contains some $f \neq 0$.

Theorem B. *Let g be a conformal map of U onto a Jordan domain whose boundary is C^∞ with positive curvature. Write $(D_g f)(z) = f'(z)/g'(z)$. Then*

(*a*) *D_g is a bounded operator $A^\infty \to A^\infty$.*

(*b*) *SA_K^∞ is D_g-invariant for any Carleson set K and any singular inner function S with $\sigma(S) \subset K$.*

(*c*) *$S_a^t A_a^\infty$ is D_g-residual for any $a \in T$ and any $t \ge 0$.*

(*d*) *No other subspace of A^∞ is D_g-residual.*

Of course, the eigenvectors $e^{\lambda g}$ of D_g give rise to invariant subspaces which are not of the form SA^∞_K. We conjecture that every D_g-invariant subspace containing no $e^{\lambda g}$ is as in (b). By [22] this is equivalent to the statement that every such invariant subspace is an ideal of the algebra A^∞.

Let $A(\Omega)$ consist of all functions which are holomorphic in Ω and continuous on $\bar{\Omega}$. Define $A^\infty(\Omega)$ as in Problem B, with $A(\Omega)$ in place of A. If g is as in Theorem B and $\Omega = g(U)$, then the composition $C_g f = f \circ g$ is a bounded linear isomorphism of $A^\infty(\Omega)$ onto A^∞ with $D_g = C_g D C_g^{-1}$. Thus, Theorem B completely describes the residual subspaces for differentiation on $A^\infty(\Omega)$. Similarly, Theorems A.1 and A.2 describe the invariant subspaces for integration on the conformally invariant Hardy space $H^p(\Omega)$.

The layout of the paper is as follows. Section 2 contains the verifications of parts (a) and (b) of all three theorems and part (c) of Theorem B. Part (c) of Theorem A.1 is proved in Section 3. In Section 4 we present in parallel the arguments for (c) of Theorem A.2 and (d) of Theorem B assuming only that $g(U)$ is Jordan with smooth enough boundary and that $d_g(\lambda) = O(|\lambda|^{-\delta})$ for some $\delta > 0$. We also show that Theorem A.1 is true under these hypotheses, regardless of whether $g(U)$ is convex. The proof that our assumption of positive curvature gives control of $d_g(\lambda)$ is in Section 5.

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2. Examples of Invariant Subspaces

It is clear that any conformal map of U onto a domain of finite area belongs to the Dirichlet space and therefore to VMOA. By the Kellogg-Warschawski theorem (see Theorems 3.5 and 3.6 of [20]), the hypotheses of Theorems A.2 and B imply respectively that $\log g' \in A$ and $\log g' \in A^\infty$. The following propositions therefore establish the easy halves of Theorems A.1, A.2, and B.

Proposition 2.1. *Let $0 < p < \infty$ and fix $a \in U$. Then*

- (a) *T_g is bounded on H^p if and only if $g \in$ BMOA.*
- (b) *T_g is compact on H^p if and only if $g \in$ VMOA.*
- (c) *When $g \in$ BMOA, $B_a^n H^p$ is T_g-invariant for $n = 1, 2, 3, \ldots$.*
- (d) *When $g \in$ VMOA, $\sigma(T_g) = \{0\}$.*

PROOF. For (a) and (b), see [4]. A function $f \in H^p$ is divisible by B_a^n if and only if f has at least n zeros at a. Clearly, $T_g f$ has at least one more zero at a than f has, and this gives (c). Part (d) is clear from (b), since T_g has no point spectrum. (More generally, Limani and Malman [15] have proved that $\sigma(T_g) = \{0\}$ if and only if g is approximable by H^∞-functions in the BMO-norm.) □

Proposition 2.2. *Let $1 \le p < \infty$, fix $a \in \bar{U}$ and let $g' \in H^q$, $1/p + 1/q = 1$. Then*

- (a) *$T_g H^p \subset A$.*
- (b) *T_g is compact on H^p, and $\sigma(T_g) = \{0\}$.*
- (c) *When $a \in T$, $S_a^t H^p$ is T_g-invariant for $t > 0$.*

PROOF. If $f(z) = \sum c_n z^n$, Hardy's inequality (applied to f') says that

$$\sum |c_n| \le |f(0)| + \pi \|f'\|_1, \tag{6}$$

where the norm is that of H^1. This immediately yields (a) and shows that $g \in A \subset$ VMOA. Because $\int_a^z = \int_0^z - \int_0^a$ on H^1, (b) follows from Proposition 2.1. For (c) we appeal to Lemma 3.1 of [7] which says that a function $f \in H^p$ is divisible by S_a^t if and only if

$$\limsup_{r\to 1-} (1-r)\log|f(ra)| \le -2t. \tag{7}$$

If $f \in S_a^t H^p$, then $fg' \in S_a^t H^1$. Therefore, fg' satisfies (7), and it is straightforward to verify that so does $\int_a^z fg'$. □

Lemma 2.3. *If S is a singular inner function, then $SA^\infty_{\sigma(S)} = A^\infty \cap SH^1$ is D-invariant in A^∞.*

PROOF. That $SA^\infty_{\sigma(S)}$ is closed and equals $A^\infty \cap SH^1$ is part of Theorem 4.7 of [22]. The differentiation invariance follows from the theorem of Caughran [12] which says that $f' \in SH^1$ whenever $f' \in H^1$ and $f \in SH^1$. □

Proposition 2.4. *If $\log g' \in A^\infty$, then*

- (a) *D_g is bounded on A^∞.*
- (b) *SA^∞_K is D_g-invariant for any Carleson set K and any singular inner function S with $\sigma(S) \subset K$.*
- (c) *$S_a^t A_a^\infty$ is D_g-residual for $a \in T$ and $t \ge 0$.*

PROOF. (a) This is trivial.

(b) Since every $f \in A \cap SH^1$ vanishes on $\sigma(S)$, Lemma 2.3 implies that

$$SA^\infty_K \subset SA^\infty_{\sigma(S)} \subset A^\infty_{\sigma(S)}.$$

Since S is holomorphic off $\sigma(S)$, it follows easily that $SA^\infty_K = SA^\infty_{\sigma(S)} \cap A^\infty_K$, an intersection of D_g-invariant subspaces.

(c) By Proposition 2.2, the operator $R_\lambda = T_g(I - \lambda T_g)^{-1}$ maps H^1 into A and $S_a^t H^1$ into $S_a^t H^1$. Since $D_g R_\lambda = I + \lambda R_\lambda$ it follows via Lemma 2.3 that R_λ maps A^∞ into A^∞ and $S_a^t A_a^\infty$ into $S_a^t A_a^\infty$. By the closed graph theorem, R_λ is bounded on A^∞.

As we just noted, $(D_g - \lambda I)R_\lambda = I$. On $S_a^t A_a^\infty$ we also have $T_g D_g = I$ and therefore $R_\lambda(D_g - \lambda I) = I$. Thus, D_g restricted to $S_a^t A_a^\infty$ has empty spectrum. □

3. Integration from an Interior Point

Fix a and g as in Theorem A.1. Let $g(a) = 0$, $\Omega = g(U)$, and $\varphi = 1/\sqrt{g'}$. It involves no loss of generality to assume that $a = 0$, since the composition $C_a f = f \circ B_a$ is a bounded linear automorphism of H^p with

$$(C_a^{-1} T_g C_a f)(z) = \int_0^z f g_a', \qquad g_a = g \circ B_a^{-1}.$$

We will begin by showing that Theorem A.1 is true if H^p is replaced by the space φH^2, normed so that $\|\varphi f\|$ equals the H^2-norm of f. This space is the isometric image of the Smirnov space $E^2(\Omega)$ under $f \to f \circ g$. See Chapter 10 of [13].

Proposition 3.1. *If g and φH^2 are as above, then*

(*a*) $1/g' \in H^\infty$ *and* $g' \in H^p$ *for some* $p > 1$.

(*b*) T_g *is compact on* φH^2 *with* $\sigma(T_g) = \{0\}$.

(*c*) *There exist* $q < \infty$ *such that* $H^q \subset \varphi H^2 \subset H^2$.

(*d*) *If* $p > 0$ *and* $q < \infty$, *then* $T_g^n H^p \subset H^q$ *for some positive integer n.*

(*e*) *The polynomials are dense in* φH^2.

PROOF. Recall that every bounded convex domain is Jordan and has rectifiable boundary. In particular, $g' \in H^1$ and $g \in A$. By Corollary 3.16 of [20],

$$g'(z) = g'(0) \exp\left\{ -\frac{1}{\pi} \int_0^{2\pi} \log(1 - e^{-it} z)\, d\beta(t) \right\} \tag{8}$$

where β is an increasing right-continuous function with $\beta(t + 2\pi) = \beta(t) + 2\pi$. Geometrically, $e^{i\beta(t)}$ is the direction of the forward half-tangent of $\partial\Omega$ at $g(e^{it})$. The first half of (*a*) is clear from (8).

A Jordan domain Ω with rectifiable boundary is called a *Lavrentiev domain* if there exists a constant $M < \infty$ such that

$$\Lambda(J(z, w)) \le M|z - w| \qquad (z, w \in \partial\Omega)$$

where $J(z, w)$ is the shorter arc of $\partial\Omega$ connecting z to w and $\Lambda(J)$ denotes the length of J. A quick way to see that bounded convex domains are Lavrentiev (pointed out to me by Eskil Rydhe) is the following. There exists $\epsilon > 0$ such that $\beta(s) - \beta(t) \le \pi - \epsilon$ whenever $t < s < t + \epsilon$. If $z = g(e^{it})$ and $w = g(e^{is})$ it follows that the forward half-tangent at z meets at some point α the backward half-tangent at w. The angle at α of the triangle $[z, \alpha, w]$ is at least ϵ, and the arc J from z to w lies in $[z, \alpha, w]$. Thus, for some M depending only on ϵ,

$$\Lambda(J) \le 2\pi \operatorname{diam} J \le M|z - w|.$$

For any conformal map g of U onto a Lavrentiev domain, Theorem 7.11 of [20] shows that g' is outer with $|g'|$ satisfying the Coifman-Fefferman A_∞ condition. We shall require only two consequences of the A_∞ condition, both of which are proved in [8]. The first is the classical fact that the second half of (*a*) holds. The second is that the operator

$$(Sf)(z) = \frac{1}{\varphi(z)} \int_0^z \varphi f'$$

is bounded on H^2. Since T_g is compact on H^2, it follows that ST_g is compact on H^2. This implies (*b*), because $T_g(\varphi f) = \varphi S T_g f$.

Parts (*c*), (*d*), and (*e*) are corollaries of (*a*), by Hölder's inequality, Theorem 1 of [4], and Beurling's theorem, respectively. □

If μ is a positive measure on $\mathbb{C}$ we will write $A^2(\mu)$ for the space of all entire functions in $L^2(\mu)$. By a theorem of Lutsenko and Yulmukhametov [17] (see also [16] and [18]), our assumption that Ω is bounded and convex implies that the transform

$$(\Psi h)(w) = \langle e^{wg}, h \rangle \tag{9}$$

takes the dual space E of φH^2 onto some $A^2(\nu)$, with

$$\frac{1}{C}\|h\|^2 \le \int |\Psi h|^2\, d\nu \le C\|h\|^2 \qquad (h \in E) \tag{10}$$

for some constant $C < \infty$. The fact that there exists a conjugate-linear isometry between E and φH^2 will be of no relevance in the following discussion, but we will use the fact that norm-closed subspaces of E are weak*-closed.

For $\lambda \in \mathbb{C}$ and entire functions F, define

$$(L_\lambda F)(w) = \frac{F(w) - F(\lambda)}{w - \lambda}.$$

Let $L = L_0$. Because of the normalization $g(a) = 0$, direct computation shows that $\Psi T_g^* = L\Psi$ on E. By (b) of Proposition 3.1 it follows that L is a compact operator on $A^2(\nu)$ with $\sigma(L) = \{0\}$. One checks easily that $L_\lambda = L(I - \lambda L)^{-1}$.

A subspace of $A^2(\mu)$ is called *backward-shift invariant* if it is an invariant subspace for every L_λ. The preceding paragraph shows that Y is T_g-invariant in φH^2 if and only if $\Psi Y^\perp$ is backward-shift invariant in $A^2(\nu)$. We will appeal to the following theorem from Section 2 of [2].

Theorem 3.2. *If μ is a positive measure such that*

- (i) *convergence in $A^2(\mu)$ implies locally uniform convergence,*
- (ii) *every $f \in A^2(\mu)$ is of finite order, and*
- (iii) *$A^2(\mu)$ contains the polynomials as a dense subspace,*

then every backward-shift invariant subspace of $A^2(\mu)$ is some $\mathcal{P}_n$, the space of all polynomials of degree at most n.

It is clear from (9) and (10) that (i) and (ii) hold for $\mu = \nu$. If m denotes normalized Lebesgue measure on the unit circle and h_n is the functional defined on φH^2 by

$$\langle f, h_n \rangle = \int f(z)\bar{z}^n \, dm(z),$$

then h_n is bounded by (c) of Proposition 3.1, and we have

$$(\Psi h_n)(w) = \sum_{k=0}^{\infty} \frac{w^k}{k!} \int g^k(z)\bar{z}^n \, dm(z).$$

Since g^k has k zeros at 0, it follows that Ψ maps $\mathcal{Q}_n = \operatorname{span}\{h_0, \ldots, h_n\}$ into $\mathcal{P}_n$. Since Ψ is one-to-one and $\dim \mathcal{Q}_n = \dim \mathcal{P}_n < \infty$ we must have $\Psi\mathcal{Q}_n = \mathcal{P}_n$. By the Hahn-Banach theorem the span of all h_n is dense in E, and (iii) follows.

By Theorem 3.2, T_g is *unicellular* on φH^2, that is, the invariant subspaces are totally ordered by set inclusion.

If T is a continuous linear operator on a topological vector space X, we will say that a chain of invariant subspaces $X_1 \supset X_2 \supset \cdots$ is *complete* if no nonzero $x \in X$ lies in every X_n and if $\dim X_{n-1}/X_n = 1$ for $n = 1, 2, 3, \ldots$, where $X_0 = X$.

Since Proposition 2.1 yields a complete chain of T_g-invariant subspaces of H^p, Theorem A.1 will be proved as soon as we show that T_g is unicellular on H^p. We will use an elementary operator-theoretic trick to transfer the unicellularity from φH^2.

Proposition 3.3. *Let T be a continuous linear operator on a topological vector space Z. Let X and Y be F-spaces continuously contained in Z, with $TX \subset X$, $TY \subset Y$. Let X_n be the X-closure of $T^n X$ and define Y_n similarly for Y. Suppose*

(*i*) $X_1 \supset X_2 \supset \cdots$ *is a complete chain of* T*-invariant subspaces of* X,

(*ii*) $Y_1 \supset Y_2 \supset \cdots$ *is a complete chain of* T*-invariant subspaces of* Y,

(*iii*) $T^k X \subset Y$ *and* $T^k Y \subset X$ *for some positive integer* k.

Then T *is unicellular on* X *if and only if* T *is unicellular on* Y.

PROOF. The closed graph theorem guarantees that T^k maps X continuously into Y and vice versa, and that T is a bounded operator on X and on Y.

Fix x_0 so that $X = \mathbb{C}x_0 \oplus X_1$. Since $TX_1 \subset X_2$, we must have $Tx_0 \notin X_2$, for otherwise $TX \subset X_2$. Writing $x_n = T^n x_0$, it follows that

$$X = \mathbb{C}x_0 \oplus \cdots \oplus \mathbb{C}x_{n-1} \oplus X_n \qquad (n = 1, 2, 3, \ldots).$$

In particular, T is injective on X.

Since X_n is T-invariant, T is well defined on X/X_n. The vectors $x_0, \ldots, x_{n-1}$ project to a basis of X/X_n in which the the matrix of T is a single (transposed) Jordan block. Thus T is unicellular on X/X_n. To show that T is unicellular on X it is therefore enough to show that every T-invariant subspace of X contains an X_n.

Assume T is unicellular on Y and let M be T-invariant in X. By (*ii*), the Y-closure of $T^k M$ equals Y_n for some n. Since T^k maps Y continuously into X, it follows that $T^{2k}M$ is X-dense in $T^k Y_n$. In particular, $T^k Y_n \subset M$. This implies that $X_{n+2k} \subset M$, because $T^{n+2k}X \subset T^{n+k}Y \subset T^k Y_n$. □

Take $Z = H(U)$, $X = H^p$, $Y = \varphi H^2$, and $T = T_g$, where $H(U)$ is the space of all holomorphic functions in U with the topology of locally uniform convergence. It is clear from Propositions 2.1 and 3.1 that all hypotheses of Proposition 3.3 are satisfied, except perhaps (*i*) and (*ii*).

If n is a positive integer and P is a polynomial, (*a*) of Proposition 3.1 shows that $B_a^n P = T_g f$ for some $f \in B_a^{n-1} H^\infty$. Therefore, the H^p-closure of $T_g(B_a^{n-1}H^p)$ is $B_a^n H^p$. It follows by induction that $X_n = B_a^n H^p$, and this gives (*i*). The same argument applies to φH^2 in place of H^p, by (*e*) of Proposition 3.1. This completes the proof of Theorem A.1.

For $0 < \alpha \leq 1$, let Λ_α be the space of all $f \in H(U)$ for which $(1 - |z|)^{1-\alpha}|f'(z)|$ is bounded. By Theorem 1 of [4], (*d*) of Proposition 3.1 is true for any holomorphic function g which belongs to some Λ_α. As a by-product of the preceding argument we therefore obtain the following result.

Proposition 3.4. *Suppose* $a \in U$, $g \in \Lambda_\alpha$, *and* $1/g' \in H^p$ *for some* $\alpha > 0$ *and all* $p < \infty$. *For* $0 < p < \infty$, *let* $Q(p)$ *be the statement that every* T_g*-invariant subspace of* H^p *is of the form* $B_a^n H^p$. *Then* $Q(p)$ *is true for some* p *only if it is true for all* p.

For the sake of completeness we also give an analogue for $a \in T$.

Proposition 3.5. *Suppose* $a \in T$, $g' \in H^\infty$, *and* $1/g' \in H^p$ *for all* $p < \infty$. *For* $1 \leq p < \infty$, *let* $Q(p)$ *be the statement that every* T_g*-invariant subspace of* H^p *is of the form* $S_a^t H^p$. *Then* $Q(p)$ *is true for some* p *only if it is true for all* p.

PROOF. Suppose $Q(p)$ holds, let $1 \leq q < \infty$, and let $M \subset H^q$ be T_g-invariant. By Proposition 2.2, $T_g H^q \subset H^p$. By hypothesis, the H^p-closure of $T_g M$ is some $S_a^t H^p$. In particular, (*i*) $T_g M \subset S_a^t H^p$ and (*ii*) $T_g(S_a^t H^p) \subset M$. By the theorem of Caughran stated in the proof of Lemma 2.3, (*i*) implies that $M \subset S_a^t H^q$. The derivative of S_a^t is ψS_a^t, where $(z - a)^2\psi(z)$ is constant. Therefore, (*ii*) shows that M contains $S_a^t P$ for any polynomial P with a double zero at a. Since $a \in T$, this makes $S_a^t H^q \subset M$. □

Of course, the above techniques can also be applied to spaces other than H^p, as long as g is sufficiently well behaved. For instance, if $a \in U$ and g is a conformal map of U onto a convex domain with $g' \in H^\infty$, it is easy to show using Proposition 3.3 that the T_g-invariant subspaces of the Bergman space $A^2 = H(U) \cap L^2(A)$, where A denotes area measure, are precisely $B_a^n A^2$ for $n = 1, 2, 3, \ldots$. See also Section 4 of [7].

4. Integration from a Boundary Point and Differentiation

In this section, $1 < p < \infty$. We fix $a \in \bar{U}$ and a conformal map g of U onto a Jordan domain Ω. The standing assumptions on g are

(*i*) $d_g(\lambda) = O(|\lambda|^{-\delta})$ as $|\lambda| \to \infty$ for some $\delta > 0$, and

(*ii*) $\log g' \in A$ or $\log g' \in A^\infty$, depending on the context.

The quantity $d_g(\lambda)$ was defined in (4) and is the infimum of the norms of all operators on H^p which satisfy one (hence all) of the three conditions listed in Proposition 4.1 below. In particular, $d_g(\lambda)$ does not depend on a. Results about operators on H^p will be presented in parallel with results about operators on A^∞. The stronger version of (*ii*) is needed only for the A^∞ results. We will prove in Section 5 that (*i*) is true under the hypotheses of Theorem A.2.

The dual spaces of H^p and A^∞ will be denoted by H^p_* and A^∞_*, respectively. The usual identifications of H^p_* and A^∞_* with spaces of functions or distributions will not be needed, but we will use the fact that norm-closed subspaces of H^p_* are weak*-closed.

For $f \in A^\infty$ and $n = 0, 1, 2, \ldots$, Hardy's inequality (6) implies that

$$\|D^n f\|_p \le \|D^n f\|_\infty \le \|D^n f\|_p + \pi \|D^{n+1} f\|_p.$$

Every D^n is a polynomial in D_g with coefficients in A^∞. The same is true if D and D_g are interchanged. Therefore, the topology defined on A^∞ in Section 1 is compatible with the seminorms $\|D_g^n f\|_p$. If $u \in A^\infty_*$ there thus exist $h_0, \ldots, h_n \in H^p_*$ such that

$$\langle f, u\rangle = \sum_{i=0}^{n} \langle D_g^i f, h_i\rangle \qquad (f \in A^\infty). \tag{11}$$

If u admits a representation of the form (11), we will say that u is *of order n*. Note that functionals of order n are also of order m for all $m > n$.

As in Section 3 we define a map Ψ on H^p_* and on A^∞_* by

$$(\Psi u)(w) = \langle e^{wg}, u\rangle.$$

We will often use $\hat{u}$ as a shorthand for Ψu. By Mergelyan's theorem, our assumption that Ω is a Jordan domain implies that the polynomials in g are dense in A and thus in H^p. Since

$$(D^n \hat{h})(0) = \langle g^n, h\rangle \qquad (h \in H^p_*)$$

it follows that Ψ is one-to-one on H^p_*, that is, $\{e^{wg} : w \in \mathbb{C}\}$ spans a dense subspace of H^p. That the same is true in A^∞ follows via the same estimates that led to (11), just like in the usual proof of the density of trigonometric polynomials in C^∞.

Write $X = \Psi H^p_*$ and $\mathcal{X} = \Psi A^\infty_*$. We norm X so that Ψ is an isometry and we topologize $\mathcal{X}$ so that Ψ is a homeomorphism. It is clear that every $F \in X$ is an entire function of exponential type. By (11) the same is true of $\mathcal{X}$. When $p = 2$ and $g(U)$ is convex, X coincides (up to equivalence of norms) with the space $A^2(\nu)$ of Section 3.

Proposition 4.1. *Fix $\lambda \in \mathbb{C}$ and let S be a bounded operator on H^p or on A^∞. Each of the following three statements about S implies the other two.*

(*i*) $\mathscr{R}(S - R_\lambda) \subset \mathbb{C} e^{\lambda g}$.

(*ii*) $(D_g - \lambda I)S = I$.

(*iii*) *For all u with $\hat{u}(\lambda) = 0$,*

$$(\Psi S^* u)(w) = \frac{\hat{u}(w)}{w - \lambda} \qquad (w \in \mathbb{C}).$$

PROOF. Recall from Section 2 that the operator $R_\lambda = T_g(I - \lambda T_g)^{-1}$ is bounded on H^p and on A^∞, and that $(D_g - \lambda I)R_\lambda = I$ on both spaces. Since $e^{\lambda g}$ spans the null space of $D_g - \lambda I$, (*i*) therefore implies (*ii*), and (*ii*) implies that

$$Se^{wg} - \frac{e^{wg}}{w - \lambda} \in \mathbb{C} e^{\lambda g} \qquad (w \in \mathbb{C}), \tag{12}$$

which is the dual formulation of (*iii*). If (12) holds, then $(D_g - \lambda I)Se^{wg} = e^{wg}$, hence $(S - R_\lambda)e^{wg} \in \mathbb{C} e^{\lambda g}$, for all $w \in \mathbb{C}$. Since the span of the e^{wg} is dense, (*i*) follows. □

Corollary 4.2.

(*a*) *If Y is T_g-invariant in H^p, then $\Psi Y^\perp$ is nearly invariant in X.*

(*b*) *If $\mathscr{Y}$ is D_g-residual in A^∞, then $\Psi \mathscr{Y}^\perp$ is nearly invariant in $\mathscr{X}$.*

PROOF. We need to show that Y and $\mathscr{Y}$ are invariant under some operators S satisfying (*iii*). For Y, take $S = R_\lambda$. For $\mathscr{Y}$, let $S : \mathscr{Y} \to \mathscr{Y}$ be the inverse of the restriction of $D_g - \lambda I$. Then $(S - R_\lambda)f \in \mathbb{C} e^{\lambda g}$ for all $f \in \mathscr{Y}$, and the Hahn-Banach theorem yields an extension of S to A^∞ with $\mathscr{R}(S - R_\lambda) \subset \mathbb{C} e^{\lambda g}$. □

One can show as in Section 3 that $\Psi Y^\perp$ is in fact backward-shift invariant if we take $g(a) = 0$, but this will be of little use to us. The role which Theorem 3.2 played in Section 3 will here be delegated to the following result, which is a more abstract version of Theorem 2.1 of [7].

Theorem 4.3. *Suppose $Y \subset H^p$ is closed and $N = \Psi Y^\perp$ is nearly invariant in X. Put*

$$N_0 = \{F \in N : wF(w) \in X\}.$$

If N_0 is dense in N, then $Y = \Theta H^p$ for some inner function Θ.

PROOF. If $Df = f'$ on X and $M_g f = gf$ on H^p, a simple computation shows that $D\Psi = \Psi M_g^*$ on H^p_*. In particular, D is bounded on X.

By our standing assumption on $d_g(\lambda)$ there exist operators S_λ with $\|S_\lambda\| = O(|\lambda|^{-\delta})$ satisfying (*iii*) of Proposition 4.1. For fixed $F \in N_0$, the terms of the differentiated Hadamard product (3) are therefore $O(|\lambda_n|^{-1-\delta}) = O(n^{-1-\delta})$ in norm, and it follows that $DF \in N$. Since N_0 is dense, N is D-invariant. In other words, Y is M_g-invariant. Since the polynomials in g are dense in A, this implies shift invariance and Beurling's theorem applies. □

To obtain information about near invariance in $\mathscr{X}$ we will combine Theorem 4.3 with an order reduction trick that should be compared to Proposition 3.1 of [6] and Proposition 11 of [3].

Proposition 4.4. *Suppose $u \in A^\infty_*$ is of order n with $\hat{u}(\lambda) = 0$. Define $v \in A^\infty_*$ by*

$$\hat{v}(w) = \frac{\hat{u}(w)}{w - \lambda} \qquad (w \in \mathbb{C}).$$

Then v is of order 0 if $n = 0$ and of order $n - 1$ if $n \geq 1$.

PROOF. By Proposition 4.1, $v = R^*_\lambda u$. The case $n = 0$ is clear. If $n \geq 1$, (11) yields α and β of order $n - 1$ with $u = \alpha + D^*_g \beta$. Because $D_g R_\lambda = I + \lambda R_\lambda$, we have $v = R^*_\lambda \alpha + (I + \lambda R^*_\lambda)\beta$, and the result follows by induction. □

Lemma 4.5.

(*a*) *If $Y \subset H^p$ is T_g-invariant and Θ is a nonconstant inner function with $Y \subset \Theta H^p$, then $\sigma(\Theta) = \{a\}$.*

(*b*) *If $\mathcal{Y} \subset A^\infty$ is D_g-residual, then $\mathcal{Y}$ has at most one common zero in $\bar{U}$.*

(*c*) *If $e^{\beta w} \in \mathcal{X}$, then $\beta = g(b)$ for some $b \in \bar{U}$.*

PROOF. (*a*) Iteration of the formula

$$\frac{d}{dz} \int_0^z F(z, \zeta)\, d\zeta = F(z, z) + \int_0^z \partial_z F(z, \zeta)\, d\zeta,$$

which holds for any F holomorphic in the bidisc, yields

$$(T_g^{n+1} f)(z) = \frac{1}{n!} \int_a^z [g(z) - g(\zeta)]^n f(\zeta) g'(\zeta)\, d\zeta \qquad (f \in H^p). \tag{13}$$

Suppose $\sigma(\Theta)$ contains some $b \neq a$. Then b is a common zero of $T_g Y$, and (13) shows that $\int_a^b g^n f g' = 0$ for $f \in Y$, $n = 0, 1, 2, \ldots$. Since the polynomials in g are dense in A, the ordinary Weierstrass approximation theorem applied on $[a, b]$ forces $Y \cap A = 0$. In particular, $T_g Y = 0$, contradicting the assumption that $Y \neq 0$.

(*b*) If a is a common zero for $\mathcal{Y}$, then the T_g corresponding to a maps $\mathcal{Y}$ into itself, since $T_g D_g = I$ on $\mathcal{Y}$ and $D_g \mathcal{Y} = \mathcal{Y}$. By the proof of (*a*), $\mathcal{Y}$ can have no other common zero.

(*c*) It is a consequence of Runge's theorem that if $\beta \notin g(\bar{U})$ there exist functions f_n in the span of $\{e^{wz} : w \in \mathbb{C}\}$ which have $f_n(\beta) = 1$ but which tend to 0 uniformly in some neighborhood of $g(\bar{U})$. If $\varphi_n = f_n \circ g$ and $\hat{u}(w) = e^{\beta w}$, it follows that $\langle \varphi_n, u \rangle = 1$ but $\varphi_n \to 0$ in A^∞, a contradiction. □

Theorem 4.6. *Suppose $1 < p < \infty$ and g is a conformal map of U onto a Jordan domain with $\log g' \in A$ and $d_g(\lambda) = O(|\lambda|^{-\delta})$. Let $Y \subset H^p$ be T_g-invariant.*

(*a*) *If $a \in U$, then $Y = B^n_a H^p$ for some $n = 1, 2, 3, \ldots$.*

(*b*) *If $a \in T$, then $Y = S^t_a H^p$ for some $t > 0$.*

Remark: We proved in Section 3 that the backward shift on the space $A^2(\nu)$ associated to a bounded convex domain Ω by the Lutsenko-Yulmukhametov theorem is unicellular if $0 \in \Omega$. Theorem 4.6 gives the same conclusion when $0 \in \partial\Omega$, with an uncountable chain of invariant subspaces rather than a countable one.

PROOF. Put $M = Y^\perp$, $N = \Psi M$, and let N_0 be as in Theorem 4.3. By Corollary 4.2, N is nearly invariant. If $F \in N$ and $F(\lambda) = 0$, we thus have $F(w) = (w - \lambda)G(w)$ for some $G \in N_0$.

Suppose some $F \neq 0$ in N has a zero. Then $N_0 \neq 0$ and $\bar{N}_0$ is nearly invariant. Theorem 4.3 yields Θ_0 with $\bar{N}_0 = \Psi Y_0^\perp$, $Y_0 = \Theta_0 H^p$. Because $Y \subset Y_0$, (a) of Lemma 4.5 shows that $Y \cap A$ has a common zero at a. Therefore, if $f \in H^p$ and $c \in \mathbb{C}$, then $T_g f + c \in Y$ only if $c = 0$. In dual terms, $E = \{T_g^* h : h \in M, \hat{h}(0) = 0\}$ is dense in $T_g^* M$. Since

$$(\Psi T_g^* h)(w) = \frac{\hat{h}(w) - e^{g(a)w}\hat{h}(0)}{w} \qquad (h \in H_*^p), \tag{14}$$

every $F \in \Psi E$ has $wF(w) \in X$. In other words, the annihilator of $T_g^* M$ satisfies the hypotheses of Theorem 4.3. By the Hahn-Banach theorem, the resulting Θ has

$$T_g(\Theta H^p) \subset Y \subset \Theta H^p. \tag{15}$$

If no $F \neq 0$ in N has a zero, then $N = \mathbb{C}e^{\alpha w}$ for some $\alpha \in \mathbb{C}$, by Hadamard's theorem. Since M is invariant under T_g^*, (14) shows that $\alpha = g(a)$ and $T_g^* M = 0$. Thus, (15) holds with $\Theta = 1$.

By (a) of Lemma 4.5, (15) forces $\Theta = B_a^k$ if $a \in U$ and $\Theta = S_a^t$ if $a \in T$, up to unimodular constants. It follows, as in the proofs of Propositions 3.4 and 3.5, that Y equals $B_a^n H^p$ or $S_a^t H^p$, where n is either k or $k + 1$. □

Theorem 4.7. *Suppose $1 < p < \infty$ and g is a conformal map of U onto a Jordan domain with $\log g' \in A^\infty$ and $d_g(\lambda) = O(|\lambda|^{-\delta})$ with respect to H^p. Let $\mathcal{Y} \subset A^\infty$ be D_g-residual. Then $\mathcal{Y} = S_a^t A_a^\infty$ for some $a \in T$ and some $t \geq 0$.*

PROOF. Let Y be the H^p-closure of $\mathcal{Y}$. Put

$$\mathcal{M} = \mathcal{Y}^\perp \subset A_*^\infty, \quad \mathcal{N} = \Psi\mathcal{M}, \quad M = Y^\perp \subset H_*^p, \quad N = \Psi M,$$

and let N_0 be as in Theorem 4.3. Note that M consists precisely of those $u \in \mathcal{M}$ which are of order 0. By Corollary 4.2, $\mathcal{N}$ is nearly invariant in $\mathcal{X}$.

Suppose some $F \neq 0$ in $\mathcal{N}$ has infinitely many zeros. Then $N_0 \neq 0$ by Proposition 4.4, and $\bar{N}_0$ is nearly invariant in X. Theorem 4.3 yields Θ_0 with $\bar{N}_0 = \Psi Y_0^\perp$, $Y_0 = \Theta_0 H^p$. Because $\mathcal{Y} \subset Y_0$ and $\sigma(\Theta_0) \neq \varnothing$, $\mathcal{Y}$ has a common zero $a \in \bar{U}$. As in the proof of (b) of Lemma 4.5, the T_g corresponding to a maps Y into itself. Since $\mathcal{Y}$ is D_g-invariant we must have $a \in T$, and thus $Y = S_a^t H^p$ by Theorem 4.6. By Lemma 2.3 it follows that $\mathcal{Y} \subset S_a^t A_a^\infty$. To prove equality, we need to show that $\langle f, u\rangle = 0$ whenever $f \in S_a^t A_a^\infty$ and $u \in \mathcal{M}$.

If $\hat{u}$ has infinitely many zeros, Proposition 4.4 shows that $\hat{u} = P\hat{h}$ for some polynomial P and some $h \in M$. Since

$$(\Psi D_g^* u)(w) = w\hat{u}(w) \qquad (u \in A_*^\infty)$$

and $P(D_g) f \in S_a^t A_a^\infty \subset Y$, it follows that $\langle f, u\rangle = \langle P(D_g) f, h\rangle = 0$.

If $\hat{u}$ has only finitely many zeros, then $\hat{u}(w) = P(w)e^{\beta w}$ for some polynomial P and some $\beta \in \mathbb{C}$. Since $\mathcal{N}$ is nearly invariant, $e^{\beta w} \in \mathcal{N}$. By (c) of Lemma 4.5, $\beta = g(b)$ for some $b \in \bar{U}$. If δ_b is the evaluation functional at b, it follows that $u = P(D_g^*)\delta_b$ and that b is a common zero for $\mathcal{Y}$. By (b) of Lemma 4.5, $b = a$, and thus $\langle f, u\rangle = 0$.

Finally, if no $F \neq 0$ in $\mathcal{N}$ has infinitely many zeros, the argument of the preceding paragraph shows that $\mathcal{N} = \{P(w)e^{g(a)w} : P \text{ polynomial}\}$ for some $a \in T$. In other words, $\mathcal{Y} = A_a^\infty$. □

5. An Integral Estimate

Let $1 < p < \infty$ and let g be a conformal map of U onto a Jordan domain Ω whose boundary is C^2 with positive curvature. Recall that such Ω are always strictly convex.

The notation $f \lesssim h$ will mean that $f \le Ch$ for some $C < \infty$ depending only on g and p. If $f \lesssim h$ and $f \gtrsim h$, we will write $f \sim h$.

We fix a positively oriented parametrization Γ of $\partial\Omega$ with $|\Gamma'| = 1$; Γ is periodic with period L, the length of $\partial\Omega$. By hypothesis, $|\Gamma''| \sim 1$. It will be convenient to use the notation

$$\Gamma_r(s) = [\Gamma(r+s) - \Gamma(r)]\bar{\Gamma}'(r).$$

Note that $\Gamma_r(0) = 0$, $\Gamma_r'(0) = 1$, and $\Gamma_r''(0) = i|\Gamma''(r)|$.

Lemma 5.1. *If $\Gamma_r = x_r + iy_r$ and*

$$\alpha_r(s) = x_r(s)y_r(s/2) - y_r(s)x_r(s/2),$$

then $y_r(s) \sim |\Gamma_r(s)|^2$ and $|\alpha_r(s)| \gtrsim |\Gamma_r(s/2)||\Gamma_r(s)|^2$.

PROOF. The geometry of Ω guarantees that $|\Gamma_r|$, y_r, and $|\alpha_r|$ are all strictly positive on $(0, L)$ and on $(L, 2L)$. For $|\Gamma_r|$ this is clear, for y_r one may use the fact that $\partial\Omega$ intersects each of its tangents only once, and for $|\alpha_r|$ it suffices to note that no line can intersect $\partial\Omega$ thrice. The required estimates now follow easily from the Taylor expansion

$$\Gamma_r(s) = s + \frac{s^2}{2}[x_r''(\xi) + iy_r''(\eta)], \qquad |\xi|, |\eta| < |s|. \qquad \square$$

Lemma 5.2. *If $w_\lambda \in \partial\Omega$ is the point that maximizes* $\operatorname{Re} \lambda w$*, then*

$$\frac{\operatorname{Re} \lambda(w_\lambda - z)}{|\lambda|} \sim |z - w_\lambda|^2 \qquad (z \in \partial\Omega,\ \lambda \neq 0).$$

PROOF. The left side is the distance from z to the tangent of $\partial\Omega$ at w_λ. If $w_\lambda = \Gamma(r)$ and $z = \Gamma(r+s)$, this equals $y_r(s) \sim |\Gamma_r(s)|^2 = |z - w_\lambda|^2$, by Lemma 5.1. $\square$

Lemma 5.3. *If $\zeta_t = (1-t)w + tz$, then*

$$1 - |g^{-1}(\zeta_t)| \gtrsim t(1-t)|z - w|^2 \qquad (z, w \in \partial\Omega,\ 0 \le t \le 1).$$

PROOF. By the Koebe distortion theorem (see Corollary 1.4 of [20]) the left side is comparable to the distance from ζ_t to $\partial\Omega$. Let a and b be the midpoints of the complementary arcs of $\{z, w\}$, so that $a = \Gamma(r + s/2)$ if $w = \Gamma(r)$ and $z = \Gamma(r+s)$, $0 < s < L$. Since Ω is convex, the triangles $[w, a, z]$ and $[z, b, w]$ lie in $\bar{\Omega}$. If E is the union of the four line segments $[w, a]$, $[a, z]$, $[z, b]$, $[b, w]$ we therefore have $d(\zeta_t, \partial\Omega) \ge d(\zeta_t, E)$. By symmetry it suffices to estimate $d(\zeta_t, [w, a]) = d(t\Gamma_r(s), [0, \Gamma_r(s/2)])$. In the notation of Lemma 5.1, the line through 0 and $\Gamma_r(s/2)$ has normal $\beta_r(s) = y_r(s/2) - ix_r(s/2)$, hence its distance from $t\Gamma_r(s)$ is

$$\frac{|\operatorname{Re} t\Gamma_r(s)\bar{\beta}_r(s)|}{|\beta_r(s)|} = \frac{t|\alpha_r(s)|}{|\Gamma_r(s/2)|} \gtrsim t|\Gamma_r(s)|^2 = t|z - w|^2. \qquad \square$$

Theorem 5.4. *If $1 < p < \infty$ and g is a conformal map of U onto a Jordan domain whose boundary is C^2 with positive curvature, then $d_g(\lambda) \lesssim |\lambda|^{-\delta}$ for some $\delta > 0$.*

PROOF. Let w_λ be as in Lemma 5.2 and define S_λ on H^p by

$$(S_\lambda f)(\psi(z)) = \int_{w_\lambda}^{z} e^{\lambda(z-\zeta)} f(\psi(\zeta))\, d\zeta, \qquad (z \in \bar{\Omega}),$$

where $\psi = g^{-1}$. Direct computation yields $(D_g - \lambda I)S_\lambda = I$, so that $d_g(\lambda) \le \|S_\lambda\|$ by Proposition 4.1.

Let $z \in \partial\Omega$, let f be a polynomial in g, and put $\zeta_t = (1-t)w_\lambda + tz$. We have

$$|(S_\lambda f)(\psi(z))| \le |z - w_\lambda| \left\{ \int_0^1 e^{r \operatorname{Re} \lambda(z-\zeta_t)}\, dt \right\}^{1/r} \left\{ \int_0^1 |f(\psi(\zeta_t))|^s\, dt \right\}^{1/s}$$

for $1 < r < \infty$, $1/r + 1/s = 1$. By Lemma 5.2,

$$\int_0^1 e^{r \operatorname{Re} \lambda(z-\zeta_t)}\, dt = \frac{1 - e^{-r \operatorname{Re} \lambda(w_\lambda - z)}}{r \operatorname{Re} \lambda(w_\lambda - z)} \lesssim \frac{1}{r|\lambda||z - w_\lambda|^2}.$$

By Lemma 5.3 and the fact that $|f(z)| \lesssim (1-|z|)^{-1/p}\|f\|_p$ for $f \in H^p$,

$$\left\{ \int_0^1 |f(\psi(\zeta_t))|^s\, dt \right\}^{1/s} \lesssim C(s,p)|z - w_\lambda|^{-2/p}\|f\|_p,$$

where

$$C(s,p) = \left\{ \int_0^1 t^{-s/p}(1-t)^{-s/p}\, dt \right\}^{1/s}.$$

In summary, for $z \in T$,

$$|(S_\lambda f)(z)| \lesssim C(s,p)|r\lambda|^{-1/r}|g(z) - w_\lambda|^{1-2/r-2/p}\|f\|_p.$$

If r is large enough, then $C(s,p) < \infty$ and $|g - w_\lambda|^{1-2/r-2/p} \in L^p$, uniformly in λ. Since the polynomials in g are dense in H^p, we are done. □

In view of Proposition 3.5, Theorems A.2 and B now follow from Theorems 4.6 and 4.7, respectively.

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