Points of Density in Metric Spaces and Measurability of Arbitrary Unions

Erik Avelin

Abstract

We prove the Lebesgue density theorem for a regular doubling measure on a metric space and show that it leads to a simple proof of the measurability of arbitrary unions of certain measurable sets. Special cases include the Lebesgue measurability of arbitrary unions of convex bodies, as well as the closedness of the density topology under arbitrary unions. We also consider a variant of the result, with a stronger conclusion of topological nature.

Introduction. It is a general intuition in analysis and probability theory that arbitrary unions H of sufficiently "thick" measurable sets Δ are measurable. Two classical theorems of this kind, for Lebesgue measure on R^n, are:

I. H is measurable if every point of each Δ is a point of density.

II. H is measurable if each Δ is convex and has nonempty interior.

The first goes back to Goffman and Waterman [2], who state it without proof. Details may be found in [4] and in Ch. 22 of [5]. Two proofs of II appear in [1].

This paper presents a very general result that contains both I and II as special cases. The proof is simpler and more elementary than those cited above.

In what follows, (X, d) will be a metric space and m a complete regular measure on a σ-algebra $\mathfrak{M}$ in X containing the Borel sets. We shall assume that each (open) ball $B(x, r)$ in X has finite positive measure, and that

$$m(B(x, 3r)) \leq cm(B(x, r)) \qquad (x \in X,\ r > 0), \tag{1}$$

for some constant c. Note that Lebesgue measure on R^n satisfies (1) if $c = 3^n$.

Our first objective is to show that almost every point of each $E \in \mathfrak{M}$ is a point of density. This means that $Q_r E \to 1$ a.e. on E as $r \to 0$, where

$$(Q_r E)(x) = \frac{m(E \cap B(x, r))}{m(B(x, r))} \qquad (x \in X,\ r > 0). \tag{2}$$

Fatou's lemma shows that the numerator in (2) is lower semicontinuous on X. Thus

$$TE = \limsup_{r \to 0} Q_r E$$

is measurable.

The main purpose of (1) is the following covering lemma, which leads to a weak-type inequality for TE.

Lemma 1. *If S is a finite collection of balls in X, then*

$$m(\cup S) \leq cm(\cup T) \tag{3}$$

for some disjoint subcollection T.

PROOF. Start with $T = \emptyset$, $S' = S$. Take a ball in S' of maximal radius, add it to T, and delete from S' all balls that intersect it. Repeat this until $S' = \emptyset$. The resulting T is clearly disjoint. Since each deleted ball lies in $B(x, 3r)$ for some $B(x, r) \in T$, (3) follows from (1). ////

Lemma 2. *For $E \in \mathfrak{M}$ and $\lambda > 0$,*

$$\lambda m\{TE > \lambda\} \le cm(E). \tag{4}$$

PROOF. Let $K \subset \{TE > \lambda\}$ be compact. Each point of K is the center of a ball B with $\lambda m(B) < m(E \cap B)$. Since finitely many B cover K, Lemma 1 gives disjoint B_i such that

$$\lambda m(K) \le c \sum \lambda m(B_i) < c \sum m(E \cap B_i) \le cm(E).$$

Since m is regular, this implies (4). ////

Theorem 1. *If $E \in \mathfrak{M}$, then $\lim_{r \to 0} Q_r E = 1$ a.e. on E.*

PROOF. Let $E^c = X - E$ denote the complement of E. Being a countable union of balls, X has σ-finite measure. Thus, there exists an open $V \supset E$ with $m(V - E) < \epsilon$. On E,

$$TE^c \le TV^c + T(V - E) = T(V - E),$$

so by Lemma 2,

$$\lambda m(\{TE^c > \lambda\} \cap E) \le \lambda m\{T(V - E) > \lambda\} < c\epsilon.$$

Since ϵ and λ are arbitrary, it follows that $TE^c = 0$ a.e. on E. This is the desired conclusion, since $Q_r E = 1 - Q_r E^c$. ////

Remark. While outer regularity is key in Theorem 1, the inner regularity of m is in fact unnecessary, although it greatly simplifies the proof of Lemma 2. In [3], Lemma 2 is obtained without this hypothesis, but the full Vitali covering theorem is used in place of Lemma 1.

The proof of Theorem 1 can be adapted to show that each $f \in L^1(m)$ has Lebesgue points a.e., at the cost of some elegance due to a measurability issue. Compare [6]. An alternative approach is in [3].

We have now reached our main result:

Theorem 2. *If H is a union of sets $\Delta \in \mathfrak{M}$ such that $T\Delta > 0$ on Δ, then $H \in \mathfrak{M}$.*

In terms of complements, this asserts the measurability of arbitrary intersections of measurable sets that contain their points of density.

The hypothesis of Theorem 2 is evidently weaker than that of I in the introduction. Moreover, it is satisfied if each Δ is a nondegenerate simplex. Since every convex set with nonempty interior is a union of such Δ, II is also a special case.

PROOF. Assume first that H is bounded. Let $M = \sup m(E)$, taken over all countable unions E of the Δ. If $m(E_n) \to M$, this supremum is attained at $E = \bigcup E_n$. For any Δ,

$$m(\Delta - E) = m(E \cup \Delta) - m(E) = 0,$$

so that

$$T\Delta \le T(\Delta - E) + TE = TE.$$

Hence $E \subset H \subset \{TE > 0\}$. By the completeness of m and Theorem 1, $H \in \mathfrak{M}$.

In the general case, the hypothesis is unaffected if each Δ is replaced by $\Delta \cap B$, where B is any ball. Thus $H \cap B \in \mathfrak{M}$. Since countably many B cover X, this implies $H \in \mathfrak{M}$. ////

The exhaustion of H with a countable subunion E also occurs in the proofs of I cited in the introduction. The first proof of II in [1] also uses countable subunions, obtained from a generalized Vitali covering theorem. The second proof is quite different. It exhibits H as a countable union of sets whose boundaries have measure 0. In addition to establishing measurability, this leads to a topological conclusion: H is the union of an open set and a meager set of measure 0. In particular, H has the Baire property. (See [5], Chap. 4.)

Our final theorem obtains this conclusion from a uniform variant of the hypothesis of Theorem 2, still weaker than that of II.

For arbitrary $\Delta \subset X$, define

$$A_r(\Delta) = \inf_{y \in \Delta} (Q_r \Delta^\circ)(y), \qquad A(\Delta) = \liminf_{r \to 0} A_r(\Delta),$$

where Δ° denotes the interior of Δ. One may call $A(\Delta)$ the *minimal angle* of Δ, with the following motivation: If m is Lebesgue measure on R^n and Δ is a simplex, or any polytope, $A(\Delta)$ is proportional to the smallest (solid) angle of Δ.

Theorem 3. *If H is a union of sets Δ with $A(\Delta) > 0$, then H is a countable union of sets E with $m(\partial E) = 0$.*

PROOF. Fix $\epsilon > 0$ and put

$$S = \left\{\Delta : A_r(\Delta) > \epsilon \text{ if } r < \epsilon\right\}, \qquad E = E(\epsilon) = \cup S.$$

Let $x \in \partial E$ and $0 < r < \epsilon$. Choose $\Delta \in S$ so that $d(x, y) < r$ for some $y \in \Delta$. Since $B(x, 2r) \supset B(y, r)$ and $B(x, 2r) \subset B(y, 3r)$, (1) yields

$$1 - (Q_{2r} \partial E)(x) \ge (Q_{2r} \Delta^\circ)(x) \ge (Q_r \Delta^\circ)(y)/c > \epsilon/c.$$

This contradicts Theorem 1 unless $m(\partial E) = 0$. Note that $H = \bigcup_{n=1}^{\infty} E(1/n)$. ////

References

[1] M. Balcerzak and A. Kharazishvili. On uncountable unions and intersections of measurable sets. *Georgian Math. J.*, 6:201–212, 1999.

[2] C. Goffman and D. Waterman. Approximately continuous transformations. *Proc. Amer. Math. Soc.*, 12:116–121, 1961.

[3] J. Heinonen. *Lectures on Analysis on Metric Spaces*. Springer-Verlag, 2001.

[4] J. Kupka and K. Prikry. The measurability of uncountable unions. *Amer. Math. Monthly*, 91:85–97, 1984.

[5] J. Oxtoby. *Measure and Category*. Springer-Verlag, second edition, 1980.

[6] W. Rudin. *Real and Complex Analysis*. McGraw-Hill, third edition, 1987.