Extremal Pluriharmonic Measures

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Let U^n denote the unit polydisc in $\mathbb{C}^n$. A real Borel measure μ on the distinguished boundary T^n is called *pluriharmonic* if its Poisson integral $P[d\mu]$ is the real part of a holomorphic function. Equivalently, μ is pluriharmonic if its mixed Fourier coefficients vanish, that is, $\hat{\mu}(k) = 0$ unless $k \leq 0$ or $k \geq 0$.

By the Krein-Milman theorem, the weak*-compact set P_n of all pluriharmonic probability measures has extreme points. For $n = 1$, they are simply the unit point masses. The problem of finding the extreme points of P_n for $n > 1$ was posed by Rudin [12] in 1970 and is still wide open. Some examples have been given by McDonald [6–9] and by Samaris [13]. Forelli has obtained a necessary condition for extremality [3], and his Proposition 3.3 of [2] points out some maps that take extreme points to extreme points. More recently, Knese [5] and Bickel-Cima-Sola [1] have studied extreme points of P_n in relation to rational inner functions.

When it comes to sufficient conditions, much of the little knowledge we have is summarized in two theorems of McDonald:

Theorem 1. *Suppose $f \in H^1(U)$, $g \in H^\infty(U)$, g is inner, and $f\bar{g}\, dm$ is a probability measure on T. (The letter m denotes Haar measure.) There then exists $\mu \in P_2$ with*

$$P[d\mu](z, \cdot) = P[f\bar{g}] + \operatorname{Re} \frac{2zf}{1 - zg} \qquad (z \in U). \tag{1}$$

This μ is an extreme point if and only if f is outer.

Theorem 2. *Suppose g is an extreme point of the unit ball of $H^\infty(U)$, $g(0) = 0$, and*

$$P[d\nu](z, \cdot) = \operatorname{Re} \frac{(1 - z)(1 - g)}{1 - zg} \qquad (z \in U). \tag{2}$$

Then ν is an extreme point of P_2.

The first of these is contained in the proof of Theorem 1 of [7]; the statement is as in [8]. The second is the main theorem of [9]. It is of particular interest because it exhibits absolutely continuous extreme points (take $|g| < 1$ a.e. on T).

Below, we shall give an argument which is much easier than those in [7] and [9], and which yields both theorems at one stroke.

We will write $M(T^n)$ and $C(U^n)$ for the spaces of all complex Borel measures on T^n and all complex continuous functions on U^n, respectively. A *lattice ideal* of $C(U^n)$ is a linear subspace I such that $h \in I$ whenever $f \in I$ and $|h| \leq |f|$.

Lemma. *For each α in some index set A, choose a lattice ideal I_α of $C(U^n)$ and a positive real-linear map Λ_α of $M(T^{n+1})$ into $C(U^n)$. Then*

$$E = \{\mu \in P_{n+1} : \Lambda_\alpha \mu \in I_\alpha \text{ for all } \alpha \in A\} \tag{3}$$

is an extreme set of P_{n+1}. In particular, the extreme points of P_{n+1} that lie in E are precisely the extreme points of E.

PROOF. Let $\mu \in E$ and suppose $\nu \in M(T^{n+1})$ has $\mu + \nu,\ \mu - \nu \in P_{n+1}$. We need to show that $\mu + \nu,\ \mu - \nu \in E$.

Fix $\alpha \in A$. Since Λ_α is positive, $\Lambda_\alpha \mu \pm \Lambda_\alpha \nu \geq 0$, that is, $|\Lambda_\alpha \nu| \leq \Lambda_\alpha \mu$. Since I_α is a lattice ideal, this implies that $\Lambda_\alpha \nu \in I_\alpha$. □

We shall construct a subclass of these extreme sets E in which all Poisson integrals resemble (1). The building blocks of our Λ_α will be the $C(U^n)$-functions

$$f_k(w) = \int_{T \times T^n} \bar{\lambda}^k P_w(\eta)\, d\mu(\lambda, \eta) \qquad (k \in Z), \tag{4}$$

which we associate to each $\mu \in M(T^{n+1})$. Here, $P_w(\eta)$ is the Poisson kernel of U^n. See [11] for the general theory. These f_k determine μ uniquely; in fact,

$$P[d\mu](z, \cdot\,) = f_0 + \sum_1^\infty (z^k f_k + \bar{z}^k f_{-k}) \qquad (z \in U). \tag{5}$$

We observe that if $\mu \in P_{n+1}$ and $k \geq 1$, then $f_k \in H^1(U^n)$, $f_{-k} = \bar{f}_k$, and $|f_k| \leq f_0$.

Fix once and for all a function $g \in H^\infty(U^n)$ with $|g| \leq 1$. If I is a lattice ideal, so is the set I^* of all $f \in C(U^n)$ with $fI \subset I$. Hence, the lemma shows that

$$Q = \left\{ \mu \in P_{n+1} :\ f_0 - \operatorname{Re} \bar{g} f_1 \in I,\ (1 - g\bar{g}) f_0 \in I, \text{ and } f_0 \in I^* \right\}$$

is an extreme set of P_{n+1}.

Proposition. *If $I \cap H^{1/2}(U^n) = 0$, then each $\mu \in Q$ has*

$$f_1 - g f_0 \in I \quad \textit{and} \quad f_{k+1} = g f_k \qquad (k \geq 1). \tag{6}$$

In particular, (5) *reduces to*

$$P[d\mu](z, \cdot\,) = f_0 + \operatorname{Re} \frac{2 z f_1}{1 - zg}. \tag{7}$$

PROOF. Fix $\mu \in Q$, and write $f \sim h$ if $h - f \in I$. It is clear from (4) that

$$|2f_k - \bar{g} f_{k+1} - g f_{k-1}| \leq 2(f_0 - \operatorname{Re} \bar{g} f_1) \quad \text{and} \quad |(1 - g\bar{g}) f_k| \leq (1 - g\bar{g}) f_0$$

for all k. These majorants lie in I, so we conclude:

$$\bar{g} f_{k+1} - f_k \sim f_k - g f_{k-1}, \quad g\bar{g} f_k \sim f_k \qquad (k \in Z). \tag{8}$$

Because $gI \subset I$, it follows that

$$f_2 - g f_1 \sim g f_1 - g^2 f_0 \sim g f_1 - g^2 \operatorname{Re} \bar{g} f_1 \sim i g^2 \operatorname{Im} \bar{g} f_1. \tag{9}$$

The inequalities $\operatorname{Re} \bar{g} f_1 \leq |\bar{g} f_1| \leq f_0$ show that $\operatorname{Re} \bar{g} f_1 \sim |\bar{g} f_1|$. It is legitimate to square both sides of this relation and obtain $(\operatorname{Im} \bar{g} f_1)^2 \sim 0$, for if $f, h \in I^*$ and $f \sim h$, then $h^2 - f^2 = (h+f)(h-f) \sim 0$. Thus (9) gives $(f_2 - g f_1)^2 \sim 0$. The hypothesis on I forces $f_2 = g f_1$. Together with the first $\sim$ in (9) and the relation $f_{k+1} = 2g f_k - g^2 f_{k-1}$ $(k \geq 2)$ coming from (8), this implies (6). □

Corollary 1. *Suppose $g \in H^\infty(U^n)$ is inner and let K be the set of all $f \in H^1(U^n)$ with $f\bar{g}\, dm \in P_n$. If $f \in K$ there exists $\mu \in P_{n+1}$ with*

$$P[d\mu](z, \cdot) = P[f\bar{g}] + \operatorname{Re} \frac{2zf}{1 - zg} \qquad (z \in U). \tag{10}$$

This μ is an extreme point if and only if f is an extreme point of K.

If $n = 1$, the extreme points of K are precisely the outer functions in K; see Ex. 3 of [6]. For $n > 1$, Corollary 2.4 of [4] shows that every separately outer function in K is an extreme point.

PROOF. It follows from (5) that (10) holds for the positive measure given by

$$\int \phi\, d\mu = \int_{T^n} \phi(\bar{g}(\eta), \eta)(f\bar{g}\, dm)(\eta) \quad \text{for } \phi \in C(T^{n+1}).$$

We need only prove that the image E of K under the affine injection $f \to \mu$ is of the form (3). Let I and J be the lattice ideals of $C(U^n)$ consisting respectively of all h whose radial limits vanish a.e. and all h for which $\{h(r\eta) : 0 < r < 1\}$ is uniformly integrable on T^n. Recall that $I \cap H^{1/2}(U^n) = 0$ and that a separately harmonic h belongs to J if and only if it is the Poisson integral of a function. Therefore, (6) and (7) show that $\{\mu \in Q : f_0 \in J\} \subset E$. The opposite inclusion involves no difficulties. □

Corollary 2. *Suppose g is extremal in the unit ball of $H^\infty(U^n)$, $\operatorname{Re} g(0) = 0$, and*

$$P[d\nu](z, \cdot) = \operatorname{Re} \frac{(1 - z)(1 - g)}{1 - zg} \qquad (z \in U).$$

Then ν is an extreme point of P_{n+1}.

PROOF. Put $h = 1 - |g|$, let C_b consist of all bounded $f \in C(U^n)$, and take $I = hC_b$. Since $|g \pm hf| \le |g| + |h| = 1$ if $|f| \le 1$, the extremality of g shows that I contains no nonzero holomorphic functions. We claim that ν is the sole element of the extreme set E consisting of all $\mu \in Q$ with $P[d\mu](1, w) = 0$ for all $w \in U^n$.

Suppose $\mu \in E$. Because $P[d\mu](e^{i\theta}, w) \ge 0$, its θ-derivative is 0 at $\theta = 0$. Hence

$$f_0 + \operatorname{Re} \frac{2f_1}{1 - g} = 0 \quad \text{and} \quad \operatorname{Im} \frac{2f_1}{(1 - g)^2} = 0.$$

This says that $2f_1 = -c(1 - g)^2$ and $f_0 = c(1 - \operatorname{Re} g)$, where $c = f_0(0) = 1$. In other words, $\mu = \nu$. Conversely, these f_0 and f_1 have $f_0 - \operatorname{Re} \bar{g} f_1 = (1 - |g|^2)(1 - \frac{1}{2} \operatorname{Re} g)$, and therefore $\nu \in E$. □

Remarks. (*a*) If a lattice ideal I contains this $f_0 - \operatorname{Re} \bar{g} f_1$, it also contains any function dominated by $1 - |g|^2$. Therefore, if g lies in the unit ball B_n of $H^\infty(U^n)$ but is not an extreme point, then every I with $\nu \in Q$ has $I \cap H^{1/2}(U^n) \ne 0$.

(*b*) Any $g \in B_n$ with $\log(1 - |g|) \notin L^1(T^n)$ is an extreme point. This is the obvious sufficient condition. The well-known converse for $n = 1$ is not true for $n = 2$, see [10].

(*c*) Because of (*b*), Theorem 3.5.3 of [11] furnishes extreme points $g \in B_n$ with $g(0) = 0$ and $|g| < 1$ a.e. on T^n. To see that the corresponding ν is absolutely continuous, write $f = P[d\nu]$ and note that $f(\cdot, \eta)$ is continuous on $\bar{U}$ for almost all $\eta \in T^n$. Since $f(z, \cdot)$ is bounded in U^n for each $z \in U$, it follows that f is the Poisson integral of its radial limits.

References

[1] Kelly Bickel, Joseph A. Cima, and Alan A. Sola. Clark measures for rational inner functions. *Michigan Math. J.*, 73(5):1021–1057, 2023.

[2] Frank Forelli. A necessary condition on the extreme points of a class of holomorphic functions. *Pacific J. Math.*, 73(1):81–86, 1977.

[3] Frank Forelli. A necessary condition on the extreme points of a class of holomorphic functions. II. *Pacific J. Math.*, 92(2):277–281, 1981.

[4] Morisuke Hasumi. Extreme points and unicity of extremum problems in H^1 on polydiscs. *Pacific J. Math.*, 44:523–535, 1973.

[5] Greg Knese. Extreme points and saturated polynomials. *Illinois J. Math.*, 63(1):47–74, 2019.

[6] John N. McDonald. Measures on the torus which are real parts of holomorphic functions. *Michigan Math. J.*, 29(3):259–265, 1982.

[7] John N. McDonald. Examples of RP-measures. *Rocky Mountain J. Math.*, 16(1):191–207, 1986.

[8] John N. McDonald. Holomorphic functions on the polydisc having positive real part. *Michigan Math. J.*, 34(1):77–84, 1987.

[9] John N. McDonald. An extreme absolutely continuous RP-measure. *Proc. Amer. Math. Soc.*, 109(3):731–738, 1990.

[10] Nathaniel R. Riesenberg. An extreme point in $H^\infty(U^2)$. *Proc. Amer. Math. Soc.*, 76(1):129–130, 1979.

[11] Walter Rudin. *Function theory in polydiscs*. W. A. Benjamin, Inc., New York-Amsterdam, 1969.

[12] Walter Rudin. Harmonic analysis in polydiscs. In *Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 2*, pages 489–493. Gauthier-Villars Éditeur, Paris, 1971.

[13] N. Samaris. The extreme points of a class of functions with positive real part. *Bull. Austral. Math. Soc.*, 44(2):253–261, 1991.