Removable sets and Hausdorff Measure

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Introduction. A closed subset E of a plane open set Ω is called *removable* if every complex-valued function which is continuous in Ω and holomorphic in $\Omega - E$ is in fact holomorphic in Ω. For example, sets with no limit points in Ω are removable, by Riemann's theorem, and so are line segments, by Morera's theorem or Schwarz' reflection principle. More generally, the range of any rectifiable curve is removable. A very simple proof of this is given after Theorem 1 below.

All of the elementary results mentioned above are special cases of Theorem 2, which states that sets of σ-finite 1-dimensional Hausdorff measure are removable. This was first proved in 1929 by Besicovitch [1]. Our treatment, which is considerably simpler than the original one, is based on techniques developed by J. Garnett in [2].

Finding a necessary and sufficient geometrical condition for removability is an open problem. For a recent overview, see Sec. 3 of [4].

In what follows, $C(\Omega)$ will denote the space of all complex-valued continuous functions in Ω, and the subspace consisting of the holomorphic functions will be denoted by $H(\Omega)$. The terms "square" and "rectangle" will refer to cartesian products of closed intervals. The set of all squares with diagonal d and vertices at Gaussian integer multiples of $d/\sqrt{2}$ will be denoted by $G(d)$.

Theorem 1. *The range γ^* of a rectifiable curve γ in Ω is removable.*

PROOF. For convenience, assume that γ has length 1. Let $f \in C(\Omega)$ be holomorphic in $\Omega - \gamma^*$. Pick a rectangle $R \subset \Omega$, let $\epsilon > 0$, and choose an integer n large enough that $|f(z) - f(w)| < \epsilon$ if $z, w \in R$ and $|z - w| < 1/n$. Since the total variation function of γ is continuous, we may split γ into n parts of length $1/n$. Each part intersects less than nine $Q \in G(1/n)$, so γ^* can be covered by $9n$ squares $Q_i \in G(1/n)$. If $z_i \in q_i = Q_i \cap R$, the Cauchy theorem shows that

$$\left|\int_{\partial R} f\right| = \left|\sum \int_{\partial q_i} f\right| = \left|\sum \int_{\partial q_i} [f - f(z_i)]\right| < 9n \cdot \epsilon \cdot 4/n = 36\epsilon.$$

Thus $\int_{\partial R} f = 0$. By Morera's theorem, $f \in H(\Omega)$. □

The key property of γ^* in the above proof is that the number of squares in $G(1/n)$ which are needed to cover γ^* is $O(n)$. Clearly, general closed sets of finite length, in the Hausdorff measure sense, do not have this property. In order to circumvent this, we shall allow the squares Q_i to have different sizes and use a different ϵ for each Q_i. The resulting sum is most naturally bounded in terms of a certain weighted Hausdorff measure. Thus, we shall require a few results about Hausdorff measures with general gauge functions. More complete treatments can be found in [3] and in Chap. III of [2].

Definition. Let ϕ be a nondecreasing function on $(0, \infty)$ with $\phi(0+) = 0$. For any set E in the plane and any $\delta > 0$, put

$$\mu_\delta(E) = \inf \sum_i \phi(\operatorname{diam} E_i),$$

where the infimum is taken over all countable covers of E by sets E_i with diam $E_i < \delta$. Since $\mu_\delta(E)$ is decreasing in δ, the limit

$$\mu(E) = \lim_{\delta \to 0} \mu_\delta(E)$$

exists and is called the *Hausdorff measure of E with gauge function ϕ*. The 1-dimensional Hausdorff measure $\Lambda(E)$ is obtained by taking $\phi(t) = t$.

We claim that μ is countably subadditive. To see this, suppose $E \subset \bigcup_1^\infty E_n$ and let $\epsilon > 0$. For any $\delta > 0$, E_n can be covered by sets E_{ni} with diam $E_{ni} < \delta$ and $\sum_i \phi(\text{diam } E_{ni}) < \mu_\delta(E_n) + 2^{-n}\epsilon$. Thus,

$$\mu_\delta(E) \le \sum_{n,\,i} \phi(\text{diam } E_{ni}) < \sum_n \mu_\delta(E_n) + \epsilon \le \sum_n \mu(E_n) + \epsilon,$$

so $\mu(E) \le \sum \mu(E_n)$. Since $\mu(\emptyset) = 0$, this shows that μ is an outer measure.

The following lemma is a special case of a standard comparison result for Hausdorff measures with different gauge functions. See Lemma 1.2 in Chap. III of [2].

Lemma 1. *If $\Lambda(E) < \infty$ and $\phi(t)/t \to 0$ as $t \to 0$, then $\mu(E) = 0$.*

PROOF. Fix M so that $\Lambda(E) < M < \infty$. Let $\epsilon > 0$ and pick $\eta > 0$ so that $\phi(t) < t\epsilon/M$ if $t < \eta$. For any $\delta > 0$, E can be covered by sets E_i with diam $E_i < \delta$ and $\sum \text{diam } E_i < M$. If $\delta < \eta$, it follows that

$$\mu_\delta(E) \le \sum \phi(\text{diam } E_i) \le \frac{\epsilon}{M} \sum \text{diam } E_i < \epsilon.$$

In other words, $\mu(E) = 0$. □

In the next lemma, $m(E)$ denotes the infimum of $\sum \phi(\text{diam } Q_i)$ taken over all countable covers of E with squares Q_i in $G = \bigcup_1^\infty G(2^{-n})$.

Lemma 2. *If $\mu(E) = 0$, then $m(E) = 0$.*

PROOF. Let $\epsilon > 0$. Since $\mu_1(E) = 0$, E can be covered by sets E_i with diam $E_i < 1$ and $\sum \phi(\text{diam } E_i) < \epsilon$. Pick positive integers n_i so that $2^{-n_i} \le \text{diam } E_i < 2^{1-n_i}$. Then E_i intersects less than sixteen squares in $G(2^{-n_i})$, so

$$m(E) \le 16 \sum \phi(2^{-n_i}) \le 16 \sum \phi(\text{diam } E_i) < 16\epsilon.$$

Thus $m(E) = 0$. □

We now have sufficient machinery to give a simple proof of the Besicovitch theorem.

Theorem 2. *Suppose E is a closed subset of a plane open set Ω. If E is a countable union of sets E_n with $\Lambda(E_n) < \infty$, then E is removable.*

PROOF. Let $f \in C(\Omega)$ be holomorphic in $\Omega - E$, pick a rectangle R in Ω and let $\epsilon > 0$. We shall use the gauge function $\phi(t) = t\omega(t)$, where $\omega(t)$ is the supremum of $|f(z) - f(w)|$ taken over all $z, w \in R$ with $|z - w| < t$. Since f is uniformly continuous on R and μ is countably subadditive, Lemma 1 shows that $\mu(E) = 0$. By Lemma 2, E can be covered by countably many squares $Q_i \in G$ such that $\sum \phi(\text{diam } Q_i) < \epsilon$. If $z_i \in q_i = R \cap Q_i$, it follows that

$$\left| \int_{\partial R} f \right| = \left| \sum \int_{\partial q_i} [f - f(z_i)] \right| \le 4 \sum \phi(\text{diam } Q_i) < 4\epsilon.$$

Thus $f \in H(\Omega)$. □

References

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